A Unified Theory for a Traffic Analysis in Product Networks

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Abstract: Instead of providing separate solutions for each individual network, a unified theory is desirable to cover the study of a class of networks. Cartesian product graphs provide a common framework to investigate the performance of several individual networks. This paper addresses communication capabilities of product networks. It presents a unified theory to evaluate the traffic intensity and the saturation level of product networks. We have theoretically computed the traffic intensity and the saturation level. Examples of product networks that have been investigated are multidimensional meshes, multidimensional tori, and r-ary n-cube networks.

1 Introduction

Interconnection networks (INs) have a very important role in the performance of parallel machines. Better INs can reduce several parallel system drawbacks such as the communication time. In the past decade several networks have been proposed and studied [Hsu93], [Lati89], [Yous90], [Tzen90], and some have been used in new massively parallel processing systems as in the CM5, Campus, KSR1 and Maspar [Rama93], [Supe91]. Some INs have improved the deficiencies of the existing ones [Kuma92], [Yous90], [Tzen90], [Lati89]. The performance of INs is usually studied using a large number of measures. These measures include the degree, diameter, average distance, partitioning capability, embedding, extendibility and traffic analysis. Diameter, average distance and traffic are generally used to evaluate the communication capability of INs.

In this paper, the class of products graphs [Hara69], which will be called *product networks*, is considered. In [Yous90], a unified theory has been developed to analyze the topological properties of product network and provide common routing, embedding and partitioning algorithms for these networks. However, the traffic analysis of these networks was not considered.

In this paper, a unified theory for the traffic analysis of product networks is presented. This will enable us to provide the traffic analysis of individual networks and study their communication capabilities. A communication model based on packet switching protocol has been used to carry out the traffic analysis of product networks, under uniform traffic assumption. Intensive simulations have been conducted to validate the proposed model for different workload, different architectures, and different network sizes. Other measures such as the average time delay, the average queue size, and the throughput are computed via extensive simulations.

This paper is organized as follows. The next section overviews the product networks. A communication model and traffic intensity are described in Section 3. Section 4 presents the communication capabilities of product networks. In Section 5 Simulation results are presented and discussed. The last section concludes the paper and gives future directions.

2 Product Networks Overview

In this section product networks are defined. It also includes the topological properties as well as a routing strategy for product networks [Yous90].

Definitions:

In graph terminology, an interconnection network is modeled by a graph where the nodes represent processor elements and edges represent communication links. Therefore, product networks are interconnection networks whose underlying topologies are cartesian product graphs. The cartesian product graph of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G = G_1 \times G_2 = G_1G_2 = (V, E)$, is a graph where the set of nodes $V = V_1 \times V_2 = \{(x_1, x_2) \mid x_1 \in V_1$ and $x_2 \in V_2\}$ and $E = \{\langle(x_1, x_2), (y_1, y_2)\rangle \mid (x_1 = y_1$ and $\langle(x_2, y_2)\rangle \in E_2)$ or $(x_2 = y_2$ and $\langle(x_1, y_1)\rangle \in E_1)\}$. In the rest of this paper, a node (x, y) in G_1G_2 will be

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denoted x_1x_2 and the cardinality of the set of nodes, V_i, of a graph $G_i = (V_i, E_i)$, will be denoted $| V_i |$.

A product network G_1G_2 can be constructed by taking $| V_2 |$ copies of G_1 and connecting every set of the V_1 corresponding nodes of these copies in G_2 graph.

The definition of product networks can be generalized to the product $G_1G_2 \ldots G_i \ldots G_n$ of n networks, $G_1, G_2, \ldots, G_i, \ldots, G_n$, where each $G_i = (V_i, E_i)$ for $1 \leq i \leq n$, where $V = V_1V_2 \ldots V_n = \{x_1x_2 \ldots x_n \mid x_1 \in V_1, x_2 \in V_2, \ldots, x_n \in V_n\}$ and $E = \{\langle x_1x_2 \ldots x_n, y_1y_2 \ldots y_n\rangle \mid$ there exists an i such that $\langle x_i, y_i\rangle \in E_i$ and for every $j \neq i$ we have $x_i = y_i\}$. The topological properties of product networks can be found in [Yous90]. In this paper three popular networks are used to validate the proposed traffic analysis model: multidimensional meshes, multidimensional torues, and r-ary n-cube Networks. The popular hypercube network can be obtained from a ${}^rQ_n$ by setting $r = 2$. Note that when $r = 2$, we have K_2. Hence, a hypercube of dimension n is the product K_2^n.

3 Traffic Analysis

3.1 Communication Model

In order to compare the communication capabilities of product networks, a communication model is needed. The proposed model is a time-slotted packet switching model where packets are of fixed size. In each time slot, a processor can simultaneously send and receive a packet. It can also generate a single packet according to a Bernoulli distribution of parameter p. That is, p is the packet generation rate of every node. Destination nodes of generated packets are selected uniformly randomly. Each node contains a single unlimited buffer to accommodate incoming packets. The packets are FIFO serviced. The links between processors are full duplex. If several packets are sent to the same processor, then a conflict occurs. In this case, we assume that one of the packets is chosen randomly to be received and the others are deferred.

3.2 Traffic Intensity

The *traffic intensity* of a node x in a network $G = (V, E)$ is defined to be the product of the load of that node by the average service time of a packet [Klei76]. Since the service time is deterministic, a processor can send a packet at each time slot. Thus the average service time is assumed to be one. Therefore, the traffic intensity is reduced to the total load of the node.

The load of a node x is the total of the locally generated load, the load destinated to this node and the load that has to transit through this node. Therefore, the load is defined as:

$$\rho(x) = \rho_{local} + \rho_{transit} + \rho_{destination}$$

where

ρ_{local} = The average number of packets generated by x per time unit.

$\rho_{transit}$ = The average number of packets transiting x per time unit.

$\rho_{destination}$ = The average number of packets destinated to x per time unit

Note that $\rho_{local} = p$, where p is the packet generation rate. Let $\rho_{external}$ be equal to the sum of $\rho_{transit}$ and $\rho_{destination}$. This load depends on the total number of paths used by the routing algorithm, denoted by t_x^G, that either end at node x or have node x as an intermediate node and the traffic sent by each node to every other node. This traffic is equal to $\frac{p}{N-1}$ since each node equally sends its locally generated traffic to every other node in the system (except to itself). The following proposition gives the value of $\rho_{external}$.

Proposition 1 *The total external load a node x can receive form other nodes in the system is:*

$$\rho_{external} = \frac{p}{N-1} t_x^G$$

We define the quantity $\tau_x^G = \frac{t_x^G}{N}$, called the *external load factor* of node x in network G, and we write $\rho_{external}$:

$$\rho_{external} = p.\frac{N}{N-1}.\tau_x^G$$

Next the traffic intensity in product networks is presented.

4 Communication Capabilities of Product Networks

Consider a product network $G = G_1G_2 = (V, E)$ of 2 networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The following lemma gives the external load factor in product networks.

Lemma 1 *Let x_1 and x_2 be two nodes in G_1 and G_2, respectively, and $\tau_1^{G_1} and \tau_2^{G_2}$ their corresponding external load factors. Then the external load factor of node x_1x_2 in $G = G_1G_2$ is:*

$$\tau_{x_1x_2}^{G_1G_2} = \tau_{x_1}^{G_1} + \tau_{x2}^{G_2}$$

Proof. From proposition 1, the external load factor of x_1x_2 in G is

$$\tau_{x_1x_2}^{G_1G_2} = \frac{t_{x_1x_2}^{G_1G_2}}{N}$$

Where $N = |V_1| . |V_2|$, i.e., the number of nodes in G_1G_2,
and $t_{x_1x_2}^{G_1G_2}$ is the total number of paths that either end at node x_1x_2 or have node x_1x_2 as an intermediate node. Furthermore, it can easily concluded that $t_{x_1x_2}^{G_1G_2}$ can be expressed as:

$$t_{x_1x_2}^{G_1G_2} = t_{x_1}^{G_1} \mid V_2 \mid + t_{x_2}^{G_2} \mid V_1 \mid$$

since we have $|V_2|$ copies of G_1 and $|V_1|$ copies of G_2. Therefore,

$$\frac{t_{x_1x_2}^{G_1G_2}}{|V_1| . |V_2|} = \frac{t_{x_1}^{G_1} \mid V_2 \mid + t_{x_2}^{G_2} \mid V_1 \mid}{|V_1| . |V_2|}$$

$$= \frac{t_{x_1}^{G_1}}{|V_1|} + \frac{t_{x_2}^{G_2}}{|V_2|} = \tau_{x_1}^{G_1} + \tau_{x2}^{G_2}.$$

∎

A corollary of this lemma for the product of n graphs is that :

$$\tau_{x_1x_2\ldots x_n}^{G_1G_2\ldots G_n} = \sum_{i=1}^{i=n} \tau_{x_i}^{G_i}$$

Theorem 1 *The traffic intensity of a node x_1x_2 in a network G_1G_2 is defined as:*

$$\rho_{x_1x_2} = p(1 + \frac{N}{N-1}(\tau_{x_1}^{G_1} + \tau_{x2}^{G_2}))$$

Proof. The proof follows from proposition 1 and lemma 1. ∎

4.1 Traffic Intensity of Product Networks

To compute the traffic intensity of product networks, we first need to determine the external load factor of their building blocks.

Theorem 2 *The external load factor of a node x in each building block is:*

$$\tau_x^{K_r} = \frac{r-1}{r}$$

$$\tau_x^{L_p} = \frac{1}{p}[(p-x)(2x+1) - (x+1)]$$

$$\tau_x^{R_p} = \frac{1}{p}.[\frac{p(p+1)}{2} - \lfloor\frac{p}{2}\rfloor^2 - \lceil\frac{p}{2}\rceil]$$

The traffic intensity of each building block can be concluded from the theorem above and theorem 1.

The traffic intensity of three popular networks, namely, r-ary n-cube networks, multidimensional meshes, and multidimensional tori is computed in the next theorem.

Theorem 3 *The traffic intensity of an r-ary n-cube ${}^rQ_n = K_r^n$ of $N = r^n$ nodes, a multidimensional mesh $L_1 \times L_2 \times \ldots \times L_n$ of $N = p_1 \times p_2 \times \ldots \times p_n$ nodes, and a multidimensional torus $R_{p_1}R_{p_2}\ldots R_{p_n}$ of $N = p_1 \times p_2 \times \ldots \times p_n$ nodes are:*

$$\rho_{x_1\ldots x_n} = p(1 + \frac{N}{N-1}(n \times \frac{r-1}{r}))$$

$$\rho_{x_1\ldots x_n} = p(1 + \frac{N}{N-1}(\sum_{i=1}^{i=n}\frac{1}{p_i} \times [(p_i - x_i)(2x_i+1) - (x_i+1)]))$$

$$\rho_{x_1\ldots x_n} = p(1 + \frac{N}{N-1}(\sum_{i=1}^{i=n}\frac{1}{p_i} \times [\frac{p_i(p_i+1)}{2} - \lfloor\frac{p_i}{2}\rfloor^2 - \lceil\frac{p_i}{2}\rceil]))$$

Proof. The proof follows from theorem 1 and theorem 2. ∎

From the theorem above, we note that in the case of r-ary n-cube and torus the traffic intensity does not dependent on the node. This is because these networks are symmetric. While the traffic depends on the node in the case of a multidimensional mesh. Throughout this paper we define the traffic intensity of a network to be the traffic intensity of a given node if the network is symmetric, otherwise it is the traffic intensity of a node which handles most of the traffic.

The traffic intensity of very popular networks such as the hypercube, the two dimensional mesh, and the two dimensional torus is computed in the next corollary. The traffic intensity of a hypercube can be obtained from that of a r-ary n-cube by taking $r = 2$. In the case of a mesh and a torus, we consider a two-dimensional network of size mn where m and n are both even. Since the mesh is not symmetric, we take the traffic intensity in the node $x = \frac{n}{2}$ and $y = \frac{m}{2}$.

Corollary 1 *The traffic intensity of a hypercube of N nodes, a mesh of $N = mn$ nodes, and a torus of $N = mn$ nodes is:*

$$\rho_{x_1\ldots x_n} = p(1 + \frac{N\log_2 N}{2(N-1)})$$

$$\rho_{x_1\ldots x_n} = p(1 + \frac{N}{2(N-1)}$$

$$\times(\frac{N-1}{m} + m \times (1 - \frac{2}{N})))$$

$$\rho_{x_1 \dots x_n} = p(1 + \frac{N}{4(N-1)} \times (\frac{N}{m} + m))$$

We would like to note that the traffic intensity of a hypercube is conform with the results obtained in [Bell92].

4.2 Communication Limits: Saturation Probability

In this subsection we would like to know how much traffic a given network can handle before it becomes highly congested. This will depend on the IN structure, the routing algorithm, and the probability p of generating internal packets in the node. So for a given network and a fixed routing strategy we would like to evaluate the probability generation rate at which the network is saturated. The network is saturated whenever its traffic intensity approaches 1. The packet generation rate p at which a network is saturated will be called the *saturation probability* and denoted p_s. Therefore, the saturation probability of a network defines a new important performance factor for INs. When the packet generation rate, p, approaches p_s, the network becomes highly congested. However, when p is greater than or equal to p_s the network is flooded and the queue sizes explode. Thus, p must always be less than p_s. Furthermore, the higher p_s is, the better the network. The value of p_s for a product network can be derived from theorem 1.

Corollary 2 *The saturation probability of a node x_1x_2 in a product network G_1G_2 is:*

$$p_s(x_1x_2) = \frac{1}{1 + \frac{N}{N-1} \times (\tau_{x_1}^{G_1} + \tau_{x2}^{G_2})}$$

The following corollary summarizes the saturation probabilities of a binary cube, a mn mesh and mn torus. The size of the three networks is assumed to be N.

Corollary 3 *The saturation of the hypercubes, meshes and tori are:*

$$p_s = \frac{1}{1 + \frac{N \log_2 N}{2(N-1)}}$$

$$p_s = \frac{1}{1 + \frac{N}{2(N-1)} \times (\frac{N-1}{m} + m \times (1 - \frac{2}{N}))}$$

$$p_s = \frac{1}{1 + \frac{N}{4(N-1)} \times (\frac{N}{m} + m)}$$

4.3 Optimal networks

The performance of INs depends on several factors. One factor is the geometry (the structure) of the network [Krus87]. Using our traffic analysis, we define the optimal network structure of a given product network. Here, the optimal structure should yield a high saturation probability. In this subsection, optimal structure for an r-ary n-cubes , an mn mesh and a mn torus is determined. This is carried out by differentiating the saturation probability equation of each network.

Theorem 4 *The saturation probability of an r-ary n-cube increases with the radix r. The maximum saturation probability for both a mesh and a torus of $N = mn$ nodes, occurs when $m = n = \sqrt{N}$(square networks).*

In [Bell92], it was shown that the 4-ary n-cube, which is also called a Cross-Cube network, outperforms the regular hypercube.

The result of the mesh and torus is of no surprise. We would like to note that the above results can also be obtained by considering other factors such as the diameter and average distance.

5 Simulation Results

In order to validate the analysis presented in the previous section, we conducted intensive simulations using hypercubes, meshes, and tori with a variety of workloads. The simulator is based on the model presented in section 3. Several statistics are generated such as the average time delay or latency of a packet, the average queue length, and the network throughput.

Due to limited space only the average time delay for 32-node binary cube, mesh and torus, is given, Figure 1. These networks become highly congested when the packet generation rate in each node reaches the saturation probability. Therefore, the higher the saturation probability is, the better the network. From Figure 1 we can also concluded that the saturation probability computed theoretically and through simulations coincide.

The relationship of a mesh structure and its saturation probability is illustrated is Figure 2. The Figure emphasizes that a square mesh always yields better performance. similar results can be derived for a torus network.

6 Conclusion

In this paper a unified theory for the traffic analysis in product networks is presented. This traffic analysis is carried out under a time-slotted packet switching communication model. Based on this model, the traffic intensity of product networks is computed. We have also introduced a performance factor called saturation probability which defines the communication limits of a given network. The saturation probability was derived from the traffic intensity of the network. It presents the packet generation rate at which the network becomes highly congested. Simulations were conducted for hypercubes, meshes, and tori for different workloads. We concluded that the higher the saturation probability is, the better the network. We have also shown that the performance of a network depends on its structure. In the case of meshes and tori, we have shown that a square network yields better performance. In the case of r-ary n-cube, a network with higher radix presents better performance. Cross-cube networks outperform binary cubes.

Finally, future works should evaluate the communication capabilities of product networks assuming other communication modes such as wormhole routing and virtual cut-through.

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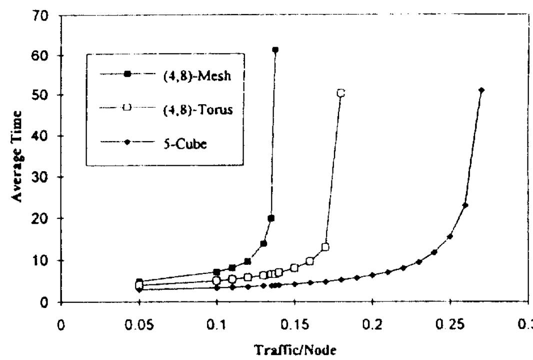

Figure 1: Network size of 32 nodes

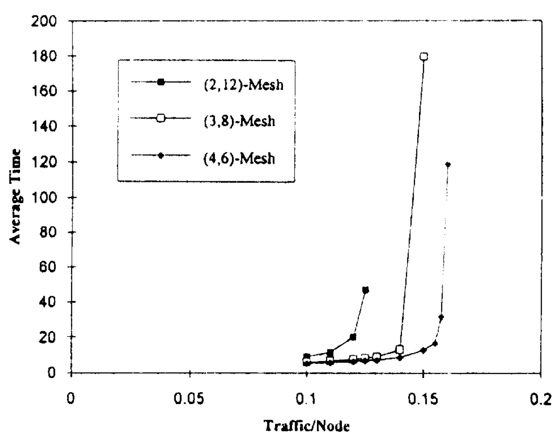

Figure 2: Mesh network with different structures